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Unifying homogeneous and inhomogeneous rheology of dense suspensions

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Abstract

The rheology of dense suspensions lacks a universal description due to the involvement of a wide variety of parameters, ranging from the physical properties of the solid particles to the nature of the external deformation or applied stress. While the former controls microscopic interactions, spatial variations in the latter induce heterogeneity in the flow, making it difficult to find suitable constitutive laws to describe the rheology in a unified way. For homogeneous driving with a spatially uniform strain rate, the rheology of non-Brownian dense suspensions is well described by the conventional $\mu(J)$ rheology. However, this rheology fails in the inhomogeneous case due to nonlocal effects, where the flow in one region is influenced by the flow in another. Here, motivated by observations from simulation data, we introduce a new dimensionless number, the suspension temperature Θ_s , which contains information on local particle velocity fluctuations. We find that $\mu(J, \Theta_s)$ provides a unified description for both homogeneous and inhomogeneous flows. By employing scaling theory, we identify a set of constitutive laws for dense suspensions of frictional spherical particles and frictionless rod-shaped particles. Combining these scaling relations with the momentum balance equation for our model system, we predict the spatial variation of the relevant dimensionless numbers, the volume fraction ϕ , the viscous number J, the macroscopic friction coefficient μ , and Θ_s solely from the nature of the imposed external driving. © 2025 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution-NonCommercial-NoDerivs 4.0 International (CC BY-NC-ND) license (https://creativecommons.org/licenses/by-nc-nd/4.0/). https://doi.org/10.1122/8.0000935

I. INTRODUCTION

Dense suspensions consist of Brownian or non-Brownian solid particles suspended in viscous fluids in roughly equal proportions. Such materials, for example, cornstarch in water, slurries, blood, etc., have widespread applications in both daily life and industry, and their flow is observed in many natural phenomena. Therefore, understanding their rheology is pivotal [1-4]. However, the rheology of these materials is extremely complex and lacks a universal description due to its dependence on various parameters such as the size, asphericity, and surface roughness of the solid particles, the nature of the suspending fluid, as well as the complexity involved in the external driving. Various constitutive models have been introduced based on defining appropriate dimensionless numbers, but their validity is often limited to specific scenarios [5-10]. Therefore, identifying suitable quantities to establish constitutive laws to describe the rheology of different types of dense suspensions under various driving conditions has been a topic of intense research [6,11–17].

Under the homogeneous scenario, where the strain rate is spatially uniform, the rheology of dense suspensions is well described by three dimensionless quantities: the solid volume fraction ϕ ; the ratio of the viscous time scale η/P to shear time scale $1/\dot{\gamma}$ known as viscous number $J = \eta \dot{\gamma}/P$; and the

ratio of shear stress σ_{xy} to normal stress P known as effective or macroscopic friction coefficient $\mu = \sigma_{xy}/P$ [6]. Here, η_{N} and $\dot{\gamma}$ are suspending the fluid viscosity and strain rate, respectively. The interdependence of these dimensionless $\frac{2}{N}$ numbers forms the constitutive laws for the homogeneous $\frac{2}{N}$ rheology of dense suspensions. However, the validity or structure of such constitutive laws might alter depending $\stackrel{\text{\tiny depending}}{=}$ on the various properties of the constituent particles. Nevertheless, in real systems, the nature of the rheology is often inhomogeneous due to the spatial variation of the strain rate, so the aforementioned dimensionless numbers are not sufficient [11]. Under homogeneous straining, J decreases with decreasing μ and increasing ϕ and, eventually, vanishes when μ and ϕ approach their limiting value μ_I and homogeneous shear jamming volume fraction ϕ_I^H , respectively. This physically signifies the cessation of the flow, however, in the case of inhomogeneous flow, one can observe $\phi > \phi_I^H$ and $\mu < \mu_I$ even for finite J due to shear-induced particle migration [18–28] and nonlocal effects [12,18,29–31].

Nonlocal phenomena in the rheology of various soft materials are studied extensively and pictured as a process where flow in regions with $\mu > \mu_J$ facilitates flow in regions with $\mu < \mu_J$ via diffusion of local fluidity of the system [32]. Such diffusion of local fluidity can be described by an inhomogeneous Helmholtz-like equation, which suggests a cooperative motion controlled by an inherent length scale [33,34]. In the context of dry granular systems, in [35], granular fluidity is macroscopically defined as $g = \dot{\gamma}/\mu$. Later, in [36], Zhang and Kamrin established the microscopic definition of fluidity, denoted as \tilde{g} , in terms of fluctuations of particle velocity, δu . \tilde{g} is uniquely determined by the local ϕ and can

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be expressed as $\tilde{g} = ga/\delta u = F(\phi)$, where *a* is the particle diameter. Such fluidity is found to be independent of ϕ at low volume fraction [37,38] but decreases with ϕ at sufficiently large volume fraction, vanishing as ϕ approaches random close packing, ϕ_{RCP} [39], irrespective of whether the flow is homogeneous or inhomogeneous. Moreover, in [14], a new dimensionless number, granular temperature $\Theta = \rho \delta u^2/DP$, is introduced. Using power-law scaling, it is demonstrated that μ , properly scaled by Θ , unifies homogeneous and inhomogeneous rheology with the inertial number *I* being the scaling variable, thus replacing the conventional $\mu(I)$ rheology by a $\mu(I, \Theta)$ rheology for inhomogeneous flows in dry granular systems. Here, *I* is the counterpart of *J* for dry granular systems and *D* is the spatial dimension.

Similarly, motivated by the concept of Θ in [14], a recent study [13] defined a new dimensionless quantity, the suspension temperature $\Theta_s = \eta \delta u/aP$. Higher values of Θ_s stand for higher mobility of the particles, thus enabling a higher degree of softness of the system from the rheological point of view. Additionally, the suspension temperature measures the competition between convective flow, driven by external forcing, and diffusive flow driven by collisions with other particles. When particle trajectories are governed mostly by collisions, their trajectories will deviate more from the affine flow, so their suspension temperature will be large. Using this novel quantity along with the other dimensionless quantities ϕ , J, and μ , a set of constitutive laws is identified that unifies the homogeneous and inhomogeneous rheology of dense suspensions of frictionless spherical particles. However, such an approach remains unexplored for frictional and aspherical particles. Interestingly, the constitutive laws for homogeneous rheology differ for frictional and aspherical particles compared to frictionless spherical particles. For frictional particles, as the particle friction coefficient μ_p increases, the sliding between particle pairs becomes increasingly restricted, imposing constraints on both rotational and translational degrees of freedom. This, in turn, leads to a reduction in ϕ_I^H [40]. With asphericity, the effect on ϕ_I^H is nonmonotonic. Specifically, for rod-shaped particles, ϕ_I^H decreases when the aspect ratio (AR = length/diameter) exceeds an intermediate value of approximately 1.5. This decrease is due to increased entanglement, or a higher number of contacts per particle, which results in inefficient random packing. However, when the aspect ratio is below this threshold, ϕ_J^H increases because the reduced entanglement allows for more efficient packing [41-45]. Therefore, the validity of Θ_s for systems of frictional and aspherical particles remains unclear and requires further investigation.

In this work, using particle-based simulations, we unify the homogeneous and inhomogeneous rheology of dense suspensions of frictional spherical particles and frictionless rodshaped particles [46–50]. In combination with the suspension temperature defined in [13] for frictionless spherical particles, along with other dimensionless numbers used to describe the homogeneous rheology, we identify new scaling relations that collapse the data of the homogeneous and inhomogeneous rheology. We find that some of the scaling relations identified here retain the same mathematical form but exhibit different exponent values across systems involving frictionless and frictional spherical particles, as well as frictionless rod-shaped particles. However, other scaling relations apply only to specific systems, highlighting intrinsic differences in their rheology. We further validate these scaling relations by demonstrating their ability to predict various dimensionless numbers in previously unexamined simulation results.

II. SIMULATION DETAILS

We simulate two systems, consisting of non-Brownian frictional spherical particles and frictionless rod-shaped particles using LAMMPS [51,52]. For the former, the system is bidisperse, with particle radii a and 1.4a mixed in equal numbers to prevent crystallization. For the latter, the rodshaped particles are created by attaching multiple spheres with appropriate overlap to achieve the desired aspect ratio. Rods constructed in this way are considered rigid bodies, where the forces acting on each sphere within the rod are collectively summed, resulting in a translational force acting on the center of mass, along with a torque relative to the center of mass. The coordinate of the center of mass and different angles (Euler angles) are updated following rigid body dynamics [53] using these forces and torques. Once the new center of mass and its orientation with respect to the origin are known, the constituent spherical bead coordinates are updated. To prepare the initial state, we start with N particles at a very small volume fraction. The orientations of these par- ℵ ticles are randomized by running Brownian dynamics, and ₹ thereafter, the system is compressed to achieve the required 8 volume fraction. In both cases, solid particles are suspended 17 in a density ρ matched viscous liquid. The simulations are $\frac{2}{3}$ performed in a periodic box with dimensions L_x , L_y , and L_z [see Fig. 1(a)]. To vary the solid volume fraction ϕ , the number of particles is adjusted while keeping the box size constant. In the case of rod-shaped particles, the volume fraction is computed by calculating the volume of each rodshaped particle. This volume is given by the total volume of the spherical beads, provided there is no overlap between the particles. If the particles overlap, the overlapping volume is subtracted from the total volume to avoid double counting. To deform the system externally, a space-dependent streaming velocity $U^{\infty}(y)$ is introduced. As a result, a solid particle experiences three different types of interactions with its surroundings. First, the drag force and torque on a particle, due to its relative motion with respect to the streaming fluid, are modeled as

$$\boldsymbol{f}_{i}^{d} = 6\pi\eta a_{i}(\boldsymbol{U}^{\infty}(\mathbf{y}) - \boldsymbol{u}_{i}), \tag{1}$$

$$\boldsymbol{\tau}_i^d = 8\pi \eta a_i^3 (\boldsymbol{\Omega}^{\infty}(\mathbf{y}) - \boldsymbol{\omega}_i). \tag{2}$$

Here, u_i and ω_i are the linear and angular velocities of the *i*th particle, and $\Omega^{\infty} = (1/2)(\nabla \times U^{\infty})$. Second, the presence of viscous fluid resists the relative motion of a pair of particles, modeled here as a hydrodynamic lubrication force [54,55]. The leading-order term of this force and torque between a pair of particles labeled as *i* and *j* with different



FIG. 1. Inhomogeneous flow of a dense suspension of frictional spherical particles. Shown here are (a) a typical configuration of the system for $\bar{\phi} = 0.60$, with the highlighted region highlighting a coarse-graining box; and the steady-state profiles in *y* of (b) the streaming velocity field $U^{\infty}(y)$ in the \hat{x} direction (solid line) and *x* component of the coarse-grained velocity field of the particles *u* (solid points); (c) the expected shear rate for a Newtonian fluid $\dot{\gamma}^{\infty} = \partial U_x^{\infty} / \partial y$ (solid line) and the measured shear rate $\dot{\gamma} = \partial u_x / \partial y$ (solid points); (d) the measured velocity fluctuations δu ; (e) the pressure *P* and the normal stresses σ_{ui} ; (f) the shear stress σ_{xy} computed from the particle interactions.

diameters is given below [56,57]:

$$\boldsymbol{f}_{i,j}^{h} \sim \begin{cases} \frac{1}{h^{c}} (\boldsymbol{u}_{i,j} \cdot \boldsymbol{n}_{i,j}) \boldsymbol{n}_{i,j} & \text{if } h_{i,j} < 10^{-3} a', \\ \frac{1}{h_{i,j}} (\boldsymbol{u}_{i,j} \cdot \boldsymbol{n}_{i,j}) \boldsymbol{n}_{i,j} & \text{if } 10^{-3} a' \leq h_{i,j} \leq 0.05 a', \\ 0 & \text{if } h_{i,j} > 0.05 a', \end{cases}$$
(3)

$$\boldsymbol{\tau}_{i,j}^{h} \sim \begin{cases} \ln\left(\frac{a_{i}}{h^{c}}\right) \left(\boldsymbol{u}_{i,j} \times \boldsymbol{n}_{i,j}\right) & \text{if } h_{i,j} < 10^{-3}a', \\ \ln\left(\frac{a_{i}}{h_{i,j}}\right) \left(\boldsymbol{u}_{i,j} \times \boldsymbol{n}_{i,j}\right) & \text{if } 10^{-3}a' \leq h_{i,j} \leq 0.05a', \\ 0 & \text{if } h_{i,j} > 0.05a'. \end{cases}$$
(4)

Here a' is the radius of the smaller particle, $h^c = 10^{-3}a'$, $u_{ij} = (u_j - u_i)$ is the relative velocity, and $h_{ij} = (a_i + a_j)$ $-|r_{ij}|$ is the shortest distance between surface of two particles, where a_i and a_j are the radius of two different types of particles. r_{ij} is distance between the centers of the two particles pointing from *i*th to *j*th particles. n_{ij} is a unit vector given by $n_{ij} = r_{ij}/|r_{ij}|$. The reason for making the lubrication force independent of the distance at small h_{ij} is to allow particles to come into direct contact. For clarity we have omitted the scalar prefactors in our expression of Eq. (4), but these are included in the model and are identical to those reported earlier [57]. Third, the contact force and torque between a pair of particles is modeled in the following way:

$$f_{i,j}^{c} = \begin{cases} k_{n}h_{i,j}\boldsymbol{n}_{i,j} - k_{t}\boldsymbol{\xi}_{i,j} & \text{if } h_{i,j} < 0, \\ 0 & \text{if } h_{i,j} > 0, \end{cases}$$
(5)

$$\boldsymbol{\tau}_{i,j}^{c} = \begin{cases} a_{i} \left(\boldsymbol{n}_{i,j} \times k_{t} \boldsymbol{\xi}_{i,j} \right) & \text{if } h_{i,j} < 0, \\ 0 & \text{if } h_{i,j} > 0. \end{cases}$$
(6)

Here, k_n and k_t are normal and tangential stiffnesses, chosen as 7×10^3 to allow for a sufficiently large time step to simulate the system over an extended period while ensuring that particle overlaps remain minimal. $\xi_{i,j}$ is the accumulated tangential displacement between particles, computed from the time they come into contact until the contact is broken. This

displacement accounts for the history dependence of the frictional force [58]. According to Coulomb's criterion, the maximum allowable tangential force for frictional particles is given by $k_i \xi_{i,i} < \mu_p k_n h_{i,j}$. We simulate four different systems of frictional spherical particles with $\mu_p = 0.1, 0.2, 0.3,$ and 0.4 and three different systems of frictionless rod-shaped particles with AR = 1.5, 2.0, and 3.0. We obtain homogeneous _N rheology data for fixed-volume systems over a suitable range of volume fraction by generating simple shear via $\sum_{y=1}^{\infty} U^{\infty}(y) = \dot{\gamma}y\hat{x}$, with y the direction of the velocity gradient and \hat{x} the unit vector along x. To keep our system in the rateindependent regime, we choose our parameters such that $\rho \dot{\gamma} a^2 / \eta \ll 1$ and $\dot{\gamma} \sqrt{\rho a^3 / k} \ll 1$ [6]. To obtain inhomogeneous flow we specify a spatially dependent liquid velocity as $U^{\infty}(y) = \kappa \sin(2\pi y/L_y)\hat{x}$ [see Fig. 1(b) and the gradient $\dot{\gamma}^{\infty}$ in Fig. 1(c)]. κ is a constant with dimension of velocity, chosen to keep $\rho \dot{\gamma} a^2 / \eta$ below 0.01 throughout to be in the overdamped regime. Such a flow field, used in many previous studies [12,59] to obtain inhomogeneous flow, can be roughly considered two oppositely moving Poiseuille flows under periodic boundary conditions. We note that inhomogeneous rheology leads to a spatially varying volume fraction; therefore, in this work, ϕ denotes the local volume fraction, while ϕ represents the mean volume fraction averaged across the entire system. We run simulations with systems containing $\mathcal{O}(10^4)$ particles, and with $\bar{\phi} = 0.48$ to 0.62 for frictional spherical particle system and 0.49 to 0.65 for frictionless rodshaped particle system. The stress tensor is computed on a per-particle basis as $\Sigma_i = \sum_j (F_{i,j}^* \otimes r_{i,j})$, counting both contact and hydrodynamic forces. To understand the difference between homogeneous and inhomogeneous rheology in our simulation setup, we need to compare the spatially variant values of J, μ , ϕ , and Θ_s obtained via inhomogeneous flow with the spatially uniform ones obtained via homogeneous flow. In order to do that we compute the variation in y of the stress and velocity fields under inhomogeneous flow, which we do by binning particle data in blocks of width a and volume $V_b = L_x a L_z$, with the per-block value

of a quantity being simply the mean of the per-particle quantities of the particles with centers lying therein. We compute the velocity fluctuation of each particle as $\delta u_i = |u_i - \bar{u}_i|$ where \bar{u} is the average velocity of all particles with centers lying in a narrow window $\pm \Delta$ (taking $\Delta = \mathcal{O}(0.1a)$) of y, and we then bin δu_i per block. To compute Θ_s we consider only y and z components of u_i to avoid any possible correlated fluctuation originating from the structure in the shear direction (i.e., \hat{x}), especially in the case of aspherical particles. The data presented here averaged over approximately 5000 configurations in the steady state. Since we are using a bidispersed system, segregation may occur. However, the rate of segregation is orders of magnitude slower than any other process in the system. As a result, the steady-state profiles of velocity, stress, and volume fraction are reached long before any measurable segregation takes place.

III. RESULTS

In Figs. 1(b)–1(f), steady-state profiles in y of the coarse-grained particle velocity u_x (flow direction), strain rate $\dot{\gamma} = \partial u_x / \partial y$, velocity fluctuations δu , pressure P (= (-1/3)Tr(Σ)) and the normal stresses, and the shear stress σ_{xy} are shown for frictional spherical particles with $\dot{\phi} = 0.60$ and $\mu_p = 0.1$, with each plotted point representing a block. The particle velocity profile and applied streaming velocity follow a similar trend, as expected, but the former is flattened at the regions of largest ϕ leading to significant deviations between $\dot{\gamma}$ and $\dot{\gamma}^{\infty}$ (= $\partial U_x^{\infty}/\partial y$). The pressure becomes spatially uniform in the steady state, and the normal stresses exhibit weak anisotropy at the regions where $\mu > \mu_J$. The shear stress follows a similar spatial variation to the shear rate.

In Fig. 2, the spatial variation of the dimensionless numbers in the steady state is presented for two different $\bar{\phi}$, close to and far from ϕ_J^H . In Fig. 2(a), the viscous number J is presented. Since the pressure is uniform in the steady state, J looks similar to $\dot{\gamma}$ shown in Fig. 1(e). Although we start our simulation from a uniform volume fraction, with straining, particles move toward the region with smaller strain rate due to normal stress σ_{yy} imbalance and accumulate there [21,22,60]. In the steady state ϕ attains a maximum at J = 0 and decreases as J increases. μ and Θ_s have a similar variation of σ_{xy} and δu , respectively.

The spatial variation of $\partial \dot{\gamma} / \partial y$, as shown in Fig. 1(e), highlights the inhomogeneous nature of the flow in our setup, as for homogeneous flow, $\partial \dot{\gamma} / \partial y = 0$ throughout. Given that the velocity profile U^{∞} follows a sinusoidal pattern [as shown in Fig. 1(b)], regions with larger J have smaller $\partial \dot{\gamma} / \partial y$, suggesting that the data from these regions might align with homogeneous (simple shear) flow data. However, regions where both J and $\partial \dot{\gamma} / \partial y$ are small are less likely to correspond to homogeneous data. These regions exhibit reduced inhomogeneity, as the entire region moves together (creeping motion), as evidenced by the flatness of the velocity profile in Fig. 1(b). The ϕ and μ here go above and below their homogeneous limit, ϕ_I^H and μ_I , respectively, and the local flow is primarily controlled by Θ_s , as we will see later. Moreover, in [13], the inhomogeneity in the flow is quantified for a setup similar to the one studied here by demonstrating a growing length scale associated with the cooperative diffusion of fluidity. This underpins the true inhomogeneous nature of the flow in our setup.

A. Rheology of frictional spherical particles

Before we go into the details of identifying scaling relations, we present the dependence of the dimensionless numbers for both homogeneous and inhomogeneous flow in Fig. 3. In Fig. 3(a), the dependence of the macroscopic friction coefficient μ on the viscous number J is shown. The black data points represent homogeneous flow, and the black solid line is the best fit using a simple power law form

$$\mu = \mu_J(\mu_p) + A(\mu_p)J^{\alpha}.$$
(7)

Similar to [40], in our study, we find μ_J exhibits strong dependence on μ_p as shown in Fig. 3(b). The inset, however, the exponent α seems to be independent of μ_p within the studied range. The other data points are for inhomogeneous flow for different mean volume fractions, $\overline{\phi}$. One can clearly see that at large *J*, where the flow is comparatively



FIG. 2. The spatial variation of the dimensionless number and the inhomogeneous nature of the flow in the steady state for $\bar{\phi} = 0.60$ (solid line) and $\bar{\phi} = 0.55$ (dashed line). Shown are (a) the viscous number $J = \eta \dot{\gamma}/P$, (b) the local volume fraction ϕ , (c) the macroscopic friction coefficient $\mu = \sigma_{xy}/P$, (d) the suspension temperature $\Theta_s = \eta \delta u/aP$, and (e) the spatial variation of $\partial \dot{\gamma}/\partial y$, where higher values of $\partial \dot{\gamma}/\partial y$ indicates a higher degree of inhomogeneity.



FIG. 3. Relations between the dimensionless control parameters for a system of frictional spherical particles with $L_x = L_z = 20a$, $L_y = 100a$, and $\mu_p = 0.2$. Shown are the relationships between the dimensionless viscous number J and (a) the macroscopic friction coefficient μ ; (b) the local volume fraction ϕ ; and (c) the suspension temperature Θ_s , for a range of mean volume fractions $\bar{\phi}$ in both homogeneous and inhomogeneous flows. In (d), the relationship between ϕ and Θ_s is shown. The Inset of (b) shows the dependence of ϕ_J^H and μ_J on μ_p . The black solid lines in (a)–(d) represent the best fits to the homogeneous data based on Eqs. (7)–(10), respectively. The inset in (d) shows the first normal stress difference $(N_1 = \sigma_{xx} - \sigma_{yy})$ and second normal stress difference $(N_1 = \sigma_{yy} - \sigma_{zz})$ as a function of *J* for $\phi = 0.60$.

homogeneous, inhomogeneous data points are superposing on the homogeneous data points. However, at comparatively small J, the effect of inhomogeneity becomes prominent manifesting as the deviation of homogeneous and inhomogeneous data. Moreover, for inhomogeneous flow, μ goes below μ_I due to the nonlocal effect.

In Fig. 3(b), the dependence of local volume fraction ϕ on J is shown. The black points for homogeneous flow are fitted with the homogeneous constitutive law [6]

$$\phi = \phi_J^H(\mu_p) - B(\mu_p)J^\beta,\tag{8}$$

where ϕ_J^H is the μ_p dependent homogeneous shear jamming volume fraction, which is the same as $\phi_{\rm RCP}$ for vanishing μ_p but decreases with increasing μ_p [see Fig. 3(b), inset and [40]]. Similar to the $\mu - J$ plot here also inhomogeneous data fall on homogeneous data for larger J but deviate at smaller J. Also, for all inhomogeneous data, the local volume fraction can be more than ϕ_J^H for finite *J*, suggesting that, unlike homogeneous $\mu(J)$ rheology, the flow is not solely controlled by μ and J.

The dependence of Θ_s on J is shown in Fig. 3(c). For both homogeneous and inhomogeneous flow, Θ_s decreases monotonically with J but at a smaller rate for inhomogeneous flow. For a fixed J inhomogeneous Θ_s is larger than the homogeneous Θ_s , suggesting a possible significant role of Θ_s in inhomogeneous flow. For homogeneous flow, Θ_s and J are related by the following power law:

$$\Theta_s = G(\mu_p) J^{\gamma}. \tag{9}$$

This power law dependence is also reported in [28]. In Fig. 3(d), the relation between ϕ and Θ_s is shown. The homogeneous data are fitted with a power law

$$\phi = \phi_J^H(\mu_p) - D(\mu_p)\Theta_s^\delta.$$
(10)

Similar to other quantities, at smaller J, inhomogeneous data deviate from homogeneous data. Interestingly, for $\phi > \phi_J^H(\mu_p), \Theta_s$ remains nonzero indicating the role of Θ_s in flow in the regions with high ϕ . For a fixed ϕ , inhomogeneous flow has higher velocity fluctuations. For homogeneous rheology, we do not observe any significant stress differences but for inhomogeneous rheology, we observe spatial variation of the normal stresses as shown in Fig. 1. In the inset of Fig. 3(d), the first normal stress difference $N_1 = \sigma_{xx} - \sigma_{yy}$ and second normal stress difference $N_2 = \sigma_{yy} - \sigma_{zz}$ are shown as functions of J.

Our first scaling relation, shown in Figs. 4(a) and 4(e), is the divergence of the relative viscosity of the suspension $(\eta_r = \mu/J)$ at the jamming volume fraction, ϕ_J , given by $\mu/J = \mathcal{F}_1^S(\phi)$. (11)

$$\mu/J = \mathcal{F}_{1a}^{S}(\phi), \tag{11}$$

with

$$\mathcal{F}_{1a}^{S}(\phi) = \eta_0(\mu_p) \left(\phi_J(\mu_p) - \phi\right)^{\nu}.$$
 (12)

Here, $v \approx 2$. For homogeneous rheology $\phi_J = \phi_I^H$ monotonically decreases from ϕ_{RCP} with increasing μ_p [40]. For inhomogeneous flow, $\phi_J = \phi_J^{IH}$ independent of μ_p and is found to be close to ϕ_{RCP} . However, this power law divergence of viscosity for inhomogeneous flow is only valid for $\phi < \phi_I^H(\mu_p)$, although the viscosity diverges at ϕ_{RCP} . η_0 is a μ_p dependent coefficient with different values for homogeneous (η_0^H) and inhomogeneous (η_0^{IH}) rheology. We find $(\eta_0^{IH}/\eta_0^H) \approx 2$ and 3 for $\mu_p = 0.1$ and 0.5, respectively. The reason to have such dependence is the following. Since some part of our inhomogeneous simulation box is strained homogeneously (at large J) data from this region fall on top of the homogeneous flow data. Thus, we have two different power laws for homogeneous and inhomogeneous flow, which start from the same point and diverge with the same exponent but at different volume fractions. Therefore, in our scaling relation, we have a prefactor that not only depends on the μ_p but also on the nature of the rheology.

However, for inhomogeneous flow with large $\overline{\phi}$, due to particle migration, the local volume fraction goes above



FIG. 4. Identified scaling relations for frictional spherical particles with two different friction coefficients $\mu_p = 0.1$ (top) and 0.5 (bottom). In (a)–(d), the collapse of homogeneous and inhomogeneous data according to scaling relations given by Eqs. (11), (13), (15), and (17) for $\mu_p = 0.1$ are presented. The solid black lines represent the scaling function (see text for details). The inset of (d) shows an additional scaling relation obtained from Eqs. (13) and (17). (e)–(h) show the same for $\mu_p = 0.5$.

 $\phi_J^H(\mu_p)$. In this regime, the scaling relation given in Eq. (11) does not apply. The inhomogeneous flow at $\phi > \phi_J^H$ is controlled by Θ_s . In Figs. 3(b) and 3(d), the homogeneous and inhomogeneous data do not follow the same trend but for fixed ϕ , in inhomogeneous flow, both J and Θ_s seem to have higher values compared to homogeneous values. This indicates that, among the regions which have same ϕ , the flow rate is higher where velocity fluctuations are higher. This correlation leads to another scaling relation presented in Figs. 4(b) and 4(f). Here, we exploit the power law dependence of J and Θ_s on ϕ given in Eqs. (8) and (10) to establish our next scaling relation

$$J/\Theta_s = \mathcal{F}^S_{1b}(\phi). \tag{13}$$

The data collapse is supported by the following form of scaling function:

$$\mathcal{F}_{1b}^{S}(\phi) = A_0^{S} - \frac{A_0^{S}}{1 + A_1^{S} (\phi_J^{IH} - \phi)^2}, \tag{14}$$

with $\phi_J^{IH} \sim \phi_{\text{RCP}}$. Here, it is important to emphasize that the mathematical form of the scaling function is chosen purely for predictive purposes, without implying any physical significance. While the form of the scaling function may suggest certain physical phenomena, these interpretations may not hold true for the actual system. For example, $\mathcal{F}_{1b}^S(\phi)$ suggests that J/Θ_s would vanish at ϕ_J^{IH} with an exponent of 2; however, this might not be accurate, as we lack data points near ϕ_J^{IH} to confirm this behavior. The primary reason for selecting this specific functional form is that it provides a good fit for data collapse within the studied range. The argument is valid for all the scaling functions used here.

Next, we focus on the power law dependence of μ and Θ_s on J given by Eqs. (7) and (9). In both cases, homogeneous

and inhomogeneous data seem scattered but for a fixed J inhomogeneous μ lie below homogeneous data, whereas Θ_s shows an opposite trend. This suggests that regions with smaller μ have higher velocity fluctuations, which maintain the flow rate. Following [13] and [14], we attempt to scale μ by Θ_s using the power law scaling. From Eqs. (7) and (9), we expect a power law scaling $\Theta_s \mu \sim J^{\alpha+\gamma}$. However, unlike dense suspensions of frictionless spherical particles in [13], this power law scaling does not result in a satisfactory data collapse at small J. We find with an adjustment of the weight of μ and Θ_s such scaling can provide us with our next scaling relation valid for a wide range of J, given below:

$$\mu^{1.4}\Theta_s^{0.6} = \mathcal{F}_2^S(J). \tag{15}$$

Here, $\mathcal{F}_2^S(J)$ is given by

$$\mathcal{F}_{2}^{S}(J) = \begin{cases} C_{1}^{S} J^{\alpha_{1}^{S}} & \text{if } J > 10^{-2}, \\ C_{2}^{S} J^{\alpha_{2}^{S}} & \text{if } 10^{-2} \ge J > 5 \times 10^{-4}, \\ C_{3}^{S} J^{\alpha_{2}^{S}} & \text{if } J \le 5 \times 10^{-4}. \end{cases}$$
(16)

The form of the scaling function depends on μ_p , with the exponents α_1^S , α_2^S , and α_3^S found to be 1, 0.85, and 0.55, and 0.9, 0.75, 0.55 for $\mu_p = 0.1$ and 0.5, respectively. This perhaps originates from the fact that the various scaling exponents are not universal but rather dependent on the friction coefficients μ_p . The data collapse is shown in Figs. 4(c) and 4(g). Our next scaling relation is based on the granular fluidity defined as $J/\mu\Theta$ (see [36]) which uniquely depends on ϕ for both homogeneous and inhomogeneous flows. The theoretical justification of such quantity is given by kinetic theory [61,62]. Similar to this an effective suspension fluidity is introduced in [13] for suspension of spherical frictionless particles. Here, we extend this to the suspension of frictional

spherical particles, which gives us

$$J/\mu^{1.1}\Theta_s = \mathcal{F}_3^S(\phi). \tag{17}$$

Here,

$$\mathcal{F}_{3}^{S}(\phi) = D_{0}^{S}\phi^{3} - \frac{D_{0}^{S}\phi^{3}}{1 + D_{1}^{S}(\phi_{J}^{IH} - \phi)^{2}}$$
(18)

shown in Figs. 4(d) and 4(h).

Thus, we effectively have three scaling relations. The second and third scaling relations, given by Eqs. (15) and (17), are valid across a wide range of volume fractions, both above and below ϕ_J^H . In contrast, the first scaling relation is divided into two parts. The first part, Eq. (11), applies for $\phi < \phi_J^H$, while the second part, Eq. (13), is valid above ϕ_J^H . Later, we will demonstrate how these scaling relations, combined with the momentum balance equation, allow us to predict all the relevant dimensionless numbers based solely on the applied fluid flow. In addition to the scaling relations discussed above, using Eqs. (13) and (17), it is possible to deduce another relation $\mu^{1.1} \sim \mathcal{F}_{1b}(\phi)/\mathcal{F}_3(\phi)$. The data collapse for $\mu_p = 0.1$ using this scaling relation is presented in the inset of (d).

B. Rheology of frictionless rod-shaped particles

For the system of frictional spherical particles, we find decoupling of homogeneous and inhomogeneous shear jamming volume fraction (i.e., $\phi_J^H < \phi_{\text{RCP}} \approx \phi_J^{IH}$) due to the frictional constraints. Similar decoupling is also expected for rod-shaped particles due to the constraints imposed by asphericity quantified by the aspect ratio. The relations between different dimensionless numbers for this system, shown in Fig. 5, are similar to those for frictional spherical particles. The homogeneous data follow the same functional form given by Eqs. (7)–(10) and are, therefore, not discussed here. Both ϕ_J^H and μ_J exhibit nonmonotonic dependence on *AR*, with an maximum value of *AR* \approx 1.5, shown in the inset of Fig. 5(b).

We further identify the scaling relations to unify the homogeneous and inhomogeneous flow of rod-shaped, frictionless particles. We find that the scaling relation $\mu/J \sim \mathcal{F}_{1a}^S(\phi)$, which works for frictionless spherical particles across the entire range of volume fractions and for frictional spherical particles within a limited range, particularly below ϕ_J^H , does not hold for frictionless rod-shaped particles. However, the scaling relation in Eq. (13) holds over a wide range of volume fractions, including both above and below the aspect ratio-dependent ϕ_J^H . This is our first scaling relation for the system of frictionless of rod-shaped particles, which can be expressed as

 $J/\Theta_s = \mathcal{F}_1^R(\phi).$

Here,

$$\mathcal{F}_{1}^{R}(\phi) = A_{0}^{R} - \frac{A_{0}^{R}}{1 + A_{1}^{R} (\phi_{J}^{IH} - \phi)^{2.4}}$$
(20)

is the best fitted form of the master curve which vanishes at ϕ_J^{IH} as shown in Figs. 6(a) and 6(d), for AR = 1.5 and 3.0. Unlike the frictional spherical system where ϕ_J^{IH} is independent of μ_p and always same as ϕ_{RCP} , for rod-shaped particles ϕ_J^{IH} are found to be different for different AR. Similar to the scaling relation given by Eq. (15) for the system of frictional spherical particles, we find such power law scaling also works for rod-shaped particles but with different exponents:

$$\mu^{1.2}\Theta_s^{0.8} = \mathcal{F}_2^R(J), \qquad (21)_{12}^{8}$$

where the best form of the scaling function $\mathcal{F}_2^R(J)$ is the following:

$$\mathcal{F}_{2}^{R}(J) = \begin{cases} C_{1}^{R} J^{\alpha_{1}^{R}} & \text{if } J > 2 \times 10^{-2}, \\ C_{2}^{R} J^{\alpha_{2}^{R}} & \text{if } 2 \times 10^{-2} \ge J > 8 \times 10^{-4}, \\ C_{3}^{R} J^{\alpha_{3}^{R}} & \text{if } J \le 8 \times 10^{-4}. \end{cases}$$
(22)



FIG. 5. The relationships between the dimensionless control parameters for a system of frictionless rod-shaped particles with $L_x = L_z = 25a$, $L_y = 80a$, and aspect ratio AR = 2.0. Shown are the relationships between the dimensionless viscous number J and (a) the macroscopic friction coefficient μ ; (b) the local volume fraction ϕ ; and (c) the suspension temperature Θ_s , for a range of mean volume fractions $\overline{\phi}$ in both homogeneous and inhomogeneous flows. In (d), the relationship between ϕ and Θ_s is shown. The inset of (b) shows the dependence of ϕ_J^H and μ_J on AR, which are in strong agreement with [42,45]. The black solid lines in (a)–(d) represent the best fits to the homogeneous data based on Eqs. (7)–(10), respectively. The inset of (c) shows schematics of rods with two aspect ratios, AR = 1.5 and 3. The inset of (d) shows the first and second normal stress difference (N_1 and N_2) as a function of J for inhomogeneous flow with similar trend to ones reported in [47].

(19)



FIG. 6. Identified scaling relations for frictionless rod-shaped particles with two different aspect ratios, AR = 1.5 and 3. In (a)–(c), the collapse of homogeneous and inhomogeneous data according to scaling relations given by Eqs. (19), (21) and (23) are shown. The solid black lines represent the scaling function (see text for details). The inset of (c) shows an additional scaling relation obtained from Eqs. (19) and (23). (d)–(f) show the same for AR = 3. The inset of (f) shows the spatial variation of angle between the axis of rod-shaped particles with different axes. Empty and solid symbols are for homogeneous and inhomogeneous flow, and solid, dotted, and dashed lines represent x, y, and z components, respectively.

As with frictional spherical particles, the scaling exponents are independent of *AR*, but the exponents α_1^R , α_2^R , and α_3^R in the master curve are found to be 1.06, 0.83, and 0.59 for *AR* = 1.5, and 0.94, 0.78, and 0.48 for *AR* = 3.0, respectively, see Figs. 6(b) and 6(e).

Similar to frictional spherical particles our third scaling relation for rod-shaped particles describes the dependence of effective suspension fluidity on the volume fraction, uniquely defined for both homogeneous and inhomogeneous rheology. We find in this system the effective suspension fluidity can be defined as $J/\mu^{0.5}\Theta_s$ and exhibits the following relation as shown in Figs. 6(c) and 6(f):

$$J/\mu^{0.5}\Theta_s = \mathcal{F}_3^R(\phi), \tag{23}$$

where

$$\mathcal{F}_{3}^{R}(\phi) = D_{0}^{R}\phi - \frac{D_{0}^{R}\phi}{1 + D_{1}^{R}(\phi_{J}^{IH} - \phi)^{2}},$$
(24)

vanishing at an *AR* dependent volume fraction ϕ_J^{IH} . An additional scaling relation obtained from Eqs. (19) and (23) is presented in the inset of Fig. 6(c). Previous studies [41,42] suggest that orientational ordering of rod-shaped particles in the direction of \mathbf{U}^{∞} is possible under straining. We further investigate this possibility in our system by measuring the angles $\langle \theta_x \rangle$, $\langle \theta_y \rangle$, and $\langle \theta_z \rangle$ between the axis of the rod-shaped particles and the **x**, **y**, and **z** axes, respectively, as shown in the inset of Fig. 6(d) for AR = 3.0. For homogeneous flow, we find that particles exhibit a tendency to align

in the flow direction, as indicated by the smaller value of $\aleph_x \langle \theta_x \rangle$ compared to $\langle \theta_y \rangle$ and $\langle \theta_z \rangle$. However, in inhomogeneous flow, they remain more or less isotropic, possibly due to particle migration and variations in the spatial derivative of \mathbf{U}^{∞} . No qualitative change is observed for different volume fraction and *AR*.

IV. PREDICTION

Our system is characterized by four dimensionless numbers and three effective scaling relations. By examining the spatial variation of just one of these dimensionless numbers, we can comprehensively capture and describe the system's rheological behavior. In our simulations, however, the only known input is the externally applied streaming velocity profile, represented by $U^{\infty}(\mathbf{y})$. Thus, to utilize the scaling relations, we must be able to compute one of the dimensionless numbers from the information of U^{∞} . To do so, considering the inertia-free momentum balance $\nabla \cdot \Sigma = -f$ per unit volume, for the *l*th segment of the simulation cell, we express the following equation:

$$N_l 6\pi \eta a_l \left[U_{x,l}^{\infty} - u_{x,l} \right] = -\left(\frac{\partial \sigma_{xy,l}}{\partial y} \right) V_b.$$
⁽²⁵⁾

Here, N_l , $U_{x,l}^{\infty}$, $u_{x,l}$, and $\sigma_{xy,l}$ represent the particle number in the block, the liquid streaming velocity at the block center, the particle velocity, and the stress averaged over the block, which has volume V_b . a_l represents a volume averaged particle radius at l with magnitude ≈ 1.2 for spheres and 1 for rod-shaped particles. The left-hand side of Eq. (25) represents the net viscous force on the particles due to fluid drag.



FIG. 7. Predictions of the spatial variation of the dimensionless quantities ϕ , J, μ , and Θ_s using the identified scaling relations and momentum balance equation against simulation data not used for the data collapse, for the system of frictional spherical particles (top) and frictionless rod-shaped particles (bottom) with $U^{\infty}(y) = \kappa \sin(2\pi y/L_y)\hat{x}$. Shown are (a) the volume fraction ϕ ; (b) the viscous number J; (c) the macroscopic friction coefficient μ ; and (d) the suspension temperature Θ_s , with predictions given by solid green lines and simulation data in red points, for $\bar{\phi} = 0.595$ and $\mu_p = 0.1$. Prediction using constitutive laws of $\mu(J)$ rheology [6] are shown using blue dashed lines. The same quantities are shown in (e)–(h) for the system of frictionless rods for $\bar{\phi} = 0.615$ and AR = 1.5.

Under inertia-free conditions this is compensated by the net stress gradient within the block. Using our dimensionless number definitions, Eq. (25) can be reformulated for the streaming velocity at *y* as

$$U_x^{\prime\infty}(\mathbf{y}) = \left[\int_0^y \frac{1}{a} J^*(\mathbf{y}') d\mathbf{y}' - \frac{2a}{9\phi(\mathbf{y})} \left(\frac{\partial \mu^*(\mathbf{y})}{\partial y} \right) \right], \tag{26}$$

where $U_x^{\prime\infty}(y) = U_x^{\infty}(y)\eta/aP$ and the asterisks indicate multiplication by $\text{sgn}(\dot{\gamma}^{\infty}(y))$. Note that *P* is uniform at steady state and $\phi(y) = (4/3)\pi a^3 N(y)/V_b$. Thus, Eq. (26) links the externally applied liquid flow field to the profiles of *J*, μ , and ϕ .

Given U_x^{∞} , we solve Eq. (26) and the scaling relations [Eqs. (11), (13), (15), and (17)] for frictional spherical particles, and Eqs. (19), (21), and (23) for frictionless rod-shaped particles] numerically by the following method. Initially, we assume a $\phi(y)$ profile, hypothesizing accumulation of particles at points where the spatial derivative of U^{∞} is zero, starting with a simple Lorentzian form $\phi(y) = \sum_{k=1}^{n_p} a_k / [(y - y_k^0)^2 + b_k^2] + \phi_0$, with mass conserved through $\overline{\phi} = (1/L_y) \int_0^{L_y} \phi(y) dy$. Here, n_p is the number of points where the first derivative of U^{∞} is zero, y_k^0 is the coordinate of such a point, and b_k is the width of the Lorentzian function centered at y_k^0 . Next, we compute J, μ , and Θ_s directly using the scaling relations, before attempting to balance Eq. (26). The imbalance in Eq. (26) indicates the accuracy of our initial guess. We refine $\phi(y)$ by adjusting ϕ_0 , a_k , and b_k until Eq. (26) is satisfied within an acceptable tolerance.

The results, shown in Fig. 7, compare predicted outcomes against previously unseen simulation data (i.e., data not used for obtaining the scaling exponents) with $\bar{\phi} = 0.595$ for frictional spherical particles, 0.615 for frictionless rod-shaped particles, and $U^{\infty}(y) = \kappa \sin(2\pi y/L_y)\hat{x}$, demonstrating the success scaling relations in predicting y profiles of ϕ , J, μ , and Θ_s . Despite the highly nonlinear nature of the scaling relations and many orders of magnitude spread of J and Θ_s , the predictions are reasonably accurate.

V. CONCLUSION

Through particle-based simulations, we establish a universal description of the flow behavior of dense suspensions, of $\frac{1}{2}$ frictional spherical and frictionless rod-shaped particles. In § addition to the standard control parameters solid volume fraction ϕ , viscous number J, and macroscopic friction coefficient μ , we introduce a novel parameter, suspension temperature Θ_s , representing velocity fluctuations, inspired by concepts from dry granular materials. Our findings reveal scaling relations among these parameters that successfully collapse data for both homogeneous and inhomogeneous flows. Using the momentum balance, we demonstrate that the characteristics of general homogeneous and inhomogeneous flow can be predicted based on the applied external force. It is important to note that in this work, we primarily focused on finding the existing scaling relations from the simulation data rather than a thorough investigation of the physical origin of them. Other choices of dimensionless quantities, for instance, based on the rotational degrees of freedom may warrant further investigation. The exponents reported here are found by using an *ad hoc* method to obtain the best data collapse. The physical origin of these exponents and the validity of the identified scaling relations in flows with more complex geometries such as hopper flow [38] remain unexplored here but represent natural and important avenues for future work, alongside extending the proposed scaling relations to situations where the gradient of the flow rate extremely large [13].

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

All the scripts used to generate the data reported in this study are available from the corresponding author upon reasonable request.

APPENDIX: NUMERICAL VALUES OF FITTING PARAMETERS

Shown in Tables I and II are the numerical values of fitting parameters used in the scaling relations reported above.

TABLE I. Values of various quantities for different particle friction coefficient μ_p .

| μ_p | ϕ^H_J | μ_J | $\phi^{I\!H}_{_J}$ | α | β |
|---------|------------|---------|--------------------|------|------|
| 0 | 0.649 | 0.13 | 0.649 | 0.44 | 0.37 |
| 0.1 | 0.625 | 0.25 | 0.654 | 0.47 | 0.43 |
| 0.2 | 0.611 | 0.31 | 0.655 | 0.45 | 0.45 |
| 0.3 | 0.607 | 0.34 | 0.652 | 0.44 | 0.44 |
| 0.5 | 0.589 | 0.38 | 0.649 | 0.43 | 0.43 |

TABLE II. Values of various quantities for different particle aspect ratio

 AR.

| AR | ϕ^H_J | μ_J | $\phi^{I\!H}_{_J}$ | α | β |
|-----|------------|---------|--------------------|------|------|
| 0 | 0.649 | 0.13 | 0.649 | 0.44 | 0.37 |
| 1.5 | 0.680 | 0.30 | 0.712 | 0.49 | 0.42 |
| 2.0 | 0.627 | 0.17 | 0.626 | 0.42 | 0.39 |
| 3.0 | 0.569 | 0.17 | 0.575 | 0.40 | 0.38 |

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